

# Private Quantum Subsystems

Tomas Jochym-O'Connor,<sup>1,2</sup> David W. Kribs,<sup>1,3</sup> Raymond Laflamme,<sup>1,2,4</sup> and Sarah Plosker<sup>3</sup>

<sup>1</sup>*Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*

<sup>2</sup>*Department of Physics & Astronomy, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*

<sup>3</sup>*Department of Mathematics & Statistics, University of Guelph, Guelph, Ontario, N1G 2W1, Canada*

<sup>4</sup>*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario, N2L 2Y5, Canada*

(Dated: August 13, 2012)

We consider a general notion of private quantum codes wherein qubits are encoded into quantum subsystems. Private quantum channels, private subspaces, and a previously considered notion of private subsystems are all captured as special cases of this general phenomena. We provide a simple example that highlights the main differences between mappings on subsystems and subspaces and show that certain classes of channels can only be private in this subsystem setting. We also set out testable conditions for deciding when a code is private for a given channel and we discuss connections with quantum error correction.

PACS numbers: 03.67.Hk; 03.67.Pp

## INTRODUCTION

The most essential primitive for private communication between two parties, Alice and Bob, in classical computation is the one-time pad. In such a scheme, the two parties share a secret key that is unknown to an external observer Eve; this key enables reliable communication by the parties as the message appears to be a random mixture of input bits from Eve's viewpoint without the key.

Private quantum codes were initially introduced as the quantum analogue of the classical one-time pad. The basic setting for a “private quantum channel” [1, 2] is as follows: Alice and Bob share a private classical key that Alice uses to inform Bob which of a set of unitary operators  $\{U_i\}$  she has used to encode her quantum state:  $\rho \mapsto U_i \rho U_i^\dagger$ . With this information in hand, Bob can decode and recover the state  $\rho$  without disturbing it. The set of unitaries  $\{U_i\}$  and the probability distribution  $\{p_i\}$  that makes up the random key which determines the encoding unitary are shared publicly. Thus, without further information, Eve's description of the system is given by the random unitary channel  $\Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger$ . By selecting certain sets of unitary operators with appropriate coefficients, the random unitary channel will provide Eve with no information about the input state.

The body of work on private quantum codes now includes a variety of other applications, with realizations both as subspaces and subsystems of Hilbert space. Private shared reference frames exploit private subspaces and subsystems that also arise from the ignorance associated with an eavesdropper's description of a system [3, 4]. Another cryptographic application is quantum secret sharing [5, 6]. If a system is mapped by a channel to  $n$  systems, distributed among  $n$  parties, one can ask how to encode quantum information into the system in such a way that any set of parties with less than  $k$  members can learn nothing about it. And the answer is that it must be encoded into subsystems that are private for the

reduction of the channel to any  $k - 1$  or fewer parties. The idea of optimality of private quantum channels, in terms of minimizing the entropy of the classical shared key, was addressed in [7]. There are also bridges between these works and quantum error correction, formalized by the complementarity results of [8]. Connections between the study of private quantum codes and the theory of operator algebras have recently been found as well [9].

In this Letter we consider a general notion of private quantum codes wherein qubits are encoded into quantum subsystems. All the previously considered instances of private quantum codes—including private quantum channels, private subspaces, and the private subsystems referenced above—are captured as special cases of this general phenomena. We consider a phase damping example throughout the presentation that highlights the main differences between mappings on subsystems and subspaces, and show that certain classes of channels can only be private in this more subtle subsystem sense. We also set out algebraic conditions that characterize privacy of a code in terms of the Kraus operators for a given quantum channel, and we discuss connections with quantum error correction.

We now describe our basic notation and nomenclature. We will introduce extra subsystem terminology later. Given a quantum system  $S$ , with (finite-dimensional and complex) Hilbert space also denoted by  $S$ , we will use customary notation such as  $\rho$ ,  $\sigma$  for density operators and  $X$ ,  $Y$  for arbitrary operators on  $S$ . The set of linear operators on  $S$  is denoted by  $\mathcal{L}(S)$ . Linear maps on  $\mathcal{L}(S)$  can be viewed as operators acting on the operator space  $\mathcal{L}(S)$ . We use the term (*quantum*) *channel* to refer to a completely positive and trace-preserving linear map  $\Phi : \mathcal{L}(S) \rightarrow \mathcal{L}(S)$ . Such maps describe (discrete) time evolution of open quantum systems in the Schrödinger picture, and can always be written in the Choi-Kraus operator-sum form  $\Phi(\rho) = \sum_i V_i \rho V_i^\dagger$  for some operators  $V_i$  in  $\mathcal{L}(S)$ . The composition of two maps will be denoted

by  $\Phi \circ \Psi(\sigma) = \Phi(\Psi(\sigma))$ .

A (linearly closed) subspace  $C$  of  $S$  is said to be a *private subspace* for  $\Phi$  if there is a density operator  $\rho_0$  on  $S$  such that  $\Phi(|\psi\rangle\langle\psi|) = \rho_0$  for all pure states  $|\psi\rangle$  in  $C$ . By linearity,  $\Phi(\rho) = \rho_0$  for all  $\rho$  in  $\mathcal{L}(C)$ . We could also consider a collection of private states not associated with a subspace of the Hilbert space, but, as in quantum error correction, we wish to allow for arbitrary superpositions of our code states and this demands the set of states considered are linearly closed.

## PRIVATE CODES FOR PHASE DAMPING

The completely depolarizing channel ( $\Phi(\rho) = \frac{1}{\dim S}I$  for all  $\rho$ ) is the simplest, concise example of a quantum channel that is private. In this case the entire Hilbert space acts as a private code for the channel, and so in order to implement such a private channel a full set of Pauli rotations must be available. This leads to a very basic question in the study of private quantum codes: Do there exist channels with fewer physical operations such that we can still encode qubits for privacy?

Perhaps the most trivial class of channels one could imagine would be the family of phase damping channels that can be applied to any qubit of a larger Hilbert space  $\mathcal{S}$  of  $n$  qubits,

$$\Lambda_i(\rho) = \frac{1}{2}(\rho + Z_i \rho Z_i), \quad \forall \rho \in \mathcal{L}(S). \quad (1)$$

A single qubit phase damping channel is not private. Yet we can ask: can composing the phase damping channel on multiple qubits yield a private subspace  $C \subseteq S$ ? Such a question is analogous to the sort of questions that have been asked in quantum error correction for some time; for example, given a set of errors that are uncorrectable on a single qubit, does there exist a larger Hilbert space such that the action of the error on the encoded Hilbert space is correctable? The answer to such a question in quantum error correction is yes, as demonstrated by the five-qubit code which corrects for arbitrary single-qubit errors, an error that would be uncorrectable if one did not have access to a larger Hilbert space to encode the quantum information into a quantum code.

We shall define the map  $\Lambda$  as the composition of the maps  $\Lambda_i$  on each of the  $n$  qubits of the state  $\rho \in S$ ,

$$\Lambda(\rho) = \Lambda_n \circ \Lambda_{n-1} \circ \cdots \circ \Lambda_1(\rho) \quad (2)$$

Equivalently one could consider the  $n$ -product map  $\Lambda_1^{\otimes n}$  of the single qubit channel  $\Lambda_1$ . For any input state  $\rho$ , this channel will decohere all off-diagonal terms in the computational basis; as such, the resulting output density matrix will be diagonal.

Consider the case when  $n = 2$ . Every output state of  $\Lambda$  has the form

$$\rho_0 = \frac{1}{4}(II + \alpha IZ + \beta ZI + \gamma ZZ), \quad (3)$$

where  $I$  and  $Z$  are the one-qubit identity and Pauli  $Z$  matrices. The goal is to find a subspace  $C$  of dimension 2 and a state  $\rho_0 \in \mathcal{L}(S)$  such that  $\Lambda(\rho) = \rho_0 \forall \rho \in \mathcal{L}(C)$ . This would show that  $\Lambda$  has a private qubit, defined by a pair of orthogonal logical states  $|0_L\rangle, |1_L\rangle$  in  $C$ .

However, one can show that such a subspace *does not* exist. In fact we can prove the following more general result, which applies to sets of commuting normal Kraus operators as well. For succinctness we leave the proof for [10].

**Theorem 1** *Let  $\Lambda$  be a channel with Kraus operators given by a set of weighted commuting unitary operators  $\{\sqrt{p_i}U_i\}$ , where  $\sqrt{p_i}$  is a scaling factor such that  $\sum_i p_i = 1$ . Then there exists no private subspace  $C$  for  $\Lambda$ .*

Is this the end of the story? This result is intuitive—at first glance it certainly does not “feel” as though we should be able to find private codes for channels such as the phase damping maps  $\Lambda = \Lambda_n \circ \cdots \circ \Lambda_2 \circ \Lambda_1$ . However, if we allow ourselves to move away from basic subspace encodings to more delicate forms of encodings and embeddings into Hilbert space, surprisingly we do discover privacy for these channels. And the answer will lead us to a general form of subsystem code.

Consider the following logically encoded qubits in 2-qubit Hilbert space:

$$\rho_L = \frac{1}{4}(II + \alpha XX + \beta YI + \gamma ZX). \quad (4)$$

This does indeed describe a single qubit encoding, as Eq. (4) describes the coordinates for a logical Bloch sphere in 2-qubit Hilbert space with logical Pauli operators given by  $X_L = XX, Y_L = YI, Z_L = ZX$ .

Now, observe that the dephasing map  $\Lambda = \Lambda_2 \circ \Lambda_1$  acting on each density operator  $\rho_L$  produces an output state that is maximally mixed; that is,  $\Lambda(\rho_L) = \frac{1}{4}II$  for all  $\rho_L$ . Thus, we see that Eq. (4) yields a private qubit “code” for the dephasing map  $\Lambda$ . However, we know from Theorem 1 that the input space cannot be a subspace. Is it a subsystem code, at least in the sense of [3, 4, 8]? The answer is no, for the same reason: as noted in the next section, the existence of such a private subsystem code immediately implies the existence of a traditional subspace code. This example motivates a new, more general type of a private subsystem.

## PRIVATE QUANTUM SUBSYSTEMS

A quantum system  $B$  is a *subsystem* of  $S$  if we can write  $S = (A \otimes B) \oplus (A \otimes B)^\perp$ . This definition is symmetric in that  $A$  is also considered a subsystem of  $S$ . The *subspaces* of  $S$  can be viewed as subsystems  $B$  for which  $A$  is one-dimensional. As before,  $\mathcal{L}(S)$  will denote the operators on  $S$ , and now a subscript such as  $\sigma_B$  means the operator belongs to  $\mathcal{L}(B)$ .

**Definition 2** Let  $\Phi : \mathcal{L}(S) \rightarrow \mathcal{L}(S)$  be a channel and let  $B$  be a subsystem of  $S$ . Then  $B$  is a private subsystem for  $\Phi$  if there is a  $\rho_0 \in \mathcal{L}(S)$  and  $\sigma_A \in \mathcal{L}(A)$  such that

$$\Phi(\sigma_A \otimes \sigma_B) = \rho_0 \quad \forall \sigma_B \quad (5)$$

In the case of a subspace ( $\dim A = 1$ ) this definition captures the basic notion of private subspaces considered previously. For a bona fide subsystem ( $\dim A > 1$ ), however, this notion of private subsystem is more general than what has been considered in [3–6, 8], which we refer to as “operator private subsystems” in this discussion. In our definition, we have weakened the requirement that Eq. (5) must hold for all  $\sigma_A$ , and only demand that it holds for a single state on  $A$ . From a channel perspective, we have removed the operator private subsystem requirement that the map  $\Phi$  needs to split into a product of maps on the individual subsystems. As opposed to operator private subsystems, we shall see below that these more general private subsystems can exist *without* the existence of private subspaces. This concept of private subsystem also generalizes the notion of “private quantum channel” to arbitrary channels—the framework presented in [1, 2] can be regarded as the special case of private subsystems for random unitary channels. And we note that while the established definition of private quantum channel does include an ancilla as above, the extra ancilla state (our  $\sigma_A$ ) and subsystem structure does not figure centrally into [1, 2]; indeed, all examples of private channels provided therein are in fact subspaces.

Returning to our phase damping example, recall the private code in the  $n = 2$  case is encoded in the logical density operators  $\rho_L = \frac{1}{4}(II + \alpha XX + \beta YY + \gamma ZX)$ , and that each of these operators produces an output state for  $\Lambda$  that is maximally mixed. It turns out that this private code can be viewed as a single qubit subsystem embedded inside two qubit space, where the ancilla operator  $\sigma_A$  in this case is the single qubit identity operator  $I_2$ ; that is, up to a unitary equivalence, the set of operators  $\rho_L$  can be seen to generate the two qubit operator algebra  $I_2 \otimes \mathbb{M}_2$ , where  $\mathbb{M}_2$  is the algebra of  $2 \times 2$  complex matrices. To see this, it is enough to show that all two qubit states  $\rho$  of the form  $\rho = \frac{1}{4}(II + \alpha XX + \beta YY + \gamma ZX)$  can be sent through appropriate unitary gates to obtain  $\rho'$  of the form  $\rho' = \frac{1}{2}I_2 \otimes \frac{1}{2}(I_2 + \alpha'X + \beta'Y + \gamma'Z)$ . Since  $I, X, Y, Z$  form a basis for  $\mathbb{M}_2$ , the claim will follow.

One can verify that an application of the inverse of the  $T = \frac{1}{\sqrt{2}}(|0\rangle(\langle 0| + \langle 1|) + i|1\rangle(\langle 0| - \langle 1|))$  gate on the first qubit, and applications of  $CNOT_{2,1}$  and  $CNOT_{1,2}$ , with the second qubit serving as the control qubit followed by the first qubit in the controlled operations, yields the desired transformation. The composition  $CNOT_{1,2}CNOT_{2,1}((T^\dagger \otimes I_2)(\cdot)(T \otimes$

$I_2))CNOT_{2,1}CNOT_{1,2}$  acts as

$$\begin{aligned} XX &\mapsto YX \mapsto ZY \mapsto IY \\ YI &\mapsto ZI \mapsto ZZ \mapsto IZ \\ ZX &\mapsto XX \mapsto IX \mapsto IX. \end{aligned}$$

Thus, we obtain  $\rho' = \frac{1}{4}(I_4 + \gamma IX + \alpha IY + \beta IZ)$ . In defining the unitary  $U = CNOT_{1,2}CNOT_{2,1} \circ (T^\dagger \otimes I_2)$ , we see the set of operators  $U\rho_L U^\dagger$  generate the algebra  $I_2 \otimes \mathbb{M}_2$ .

Evidently this example fits into the framework of the above definition. Moreover, as opposed to operator private subsystems, by Theorem 1 this private code exists in the absence of any private subspaces. Below we also show this private code has no complementary error-correcting counterpart, a further deviation from the operator private subsystem framework [8].

### Testable Conditions

If we are given a quantum channel  $\Phi(\rho) = \sum_i V_i \rho V_i^\dagger$  and a subsystem  $B$ , we can ask if it is possible to decide whether  $B$  is private for  $\Phi$ ; and more to the point, we can ask if this can be answered in terms of the Kraus operators  $V_i$  for the channel. The analogous question in quantum error correction is answered by the fundamental Knill-Laflamme conditions [11], which provide an explicit set of algebraic constraints in terms of the Kraus operators and the code, and allow one to test whether a given code is correctable for a channel. The generalization of these conditions to the case of operator error-correcting subsystems was established in [12–14].

The following result answers this question for private quantum subsystems. In addition to the Kraus operators, we would expect such an algebraic characterization to include pieces of the fixed  $A$  state and output state  $\rho_0$  in the description—observe that this information is indeed included in the conditions.

**Theorem 3** *A subsystem  $B$  is private for a channel  $\Phi(\rho) = \sum_i V_i \rho V_i^\dagger$  with fixed  $A$  state  $\sigma_A$  and output state  $\rho_0$  if and only if there are complex scalars  $\lambda_{ijkl}$  forming an isometry  $\lambda$  such that  $\sqrt{p_k}V_j|\psi_{A,k}\rangle = \sum_{i,l} \lambda_{ijkl}\sqrt{q_l}|\phi_l\rangle\langle\psi_{B,i}|$ , where  $|\psi_{A,k}\rangle$  ( $p_k$ ) and  $|\phi_l\rangle$  ( $q_l$ ) are eigenstates (eigenvalues) of  $\sigma_A$  and  $\rho_0$  respectively,  $|\psi_{B,i}\rangle$  is an orthonormal basis for  $B$ , and where  $|\psi_{A,k}\rangle$  is viewed as a channel from  $B$  into  $S$ .*

The key observation in establishing this result is that the left and right hand sides of Eq. (5) each define channels from  $B$  to  $S$  which are in fact the same. One can then use basic results from the theory of completely positive maps to obtain the equations spelled out in the theorem. Much more can be said on this topic; in particular this result can be strengthened in special cases such as the

case of private subspaces or operator private subsystems. We leave a more detailed investigation for [10].

Here we simply point out how the 2-qubit phase damping example  $\Lambda$  decomposes in this way. In that case both  $A$  and  $B$  are both spanned by  $\{|0\rangle, |1\rangle\}$ . The eigenstates of  $\rho_0 = \frac{1}{4}I_4$  are  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ , each having eigenvalue  $\frac{1}{4}$ . The eigenstates of  $\sigma_A = \frac{1}{2}I_2$  are  $\{|\psi_{A,k}\rangle\} = \{|0\rangle, |1\rangle\}$ , with corresponding eigenvalues  $\frac{1}{2}$ . For brevity, we omit the matrix calculations for the operators  $V_j|\psi_{A,k}\rangle$ ; we note that they are  $4 \times 2$  matrices formed with Pauli and  $2 \times 2$  zero blocks. The scalar-valued matrix  $\lambda = (\lambda_{ijkl})$  is indeed an isometry. Furthermore, because the number of operators  $V_j|\psi_{A,k}\rangle$  agrees with the number of operators  $|\phi_l\rangle\langle\psi_{B,i}|$  (namely, 8), the matrix  $\lambda$  is in fact unitary.

### Failure of complementarity with QEC

Numerous links have been made between aspects of quantum error correction, quantum cryptography, and quantum key distribution. In the case of operator private subsystems and error-correcting subsystems, the complementarity theorem of [8] discussed below establishes an algebraic bridge between the two fields. This firmly links the operator quantum error correction theory to that of operator private subsystems—results in one field can immediately be exported to the other. Thus, it is natural to ask whether such a result holds in this more general setting. To discuss this we first need the concept of complementary pairs of channels.

As a consequence of the Stinespring dilation theorem [15], every channel  $\Phi$  may be seen to arise from an environment Hilbert space  $E$ , a pure state  $|\psi\rangle$  on the environment, and a unitary operator  $U$  on the composite  $S \otimes E$  in the following sense:  $\Phi(\rho) = \text{Tr}_E(U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger)$ . Tracing out the system instead yields a complementary channel:  $\Phi^\sharp(\rho) = \text{Tr}_S(U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger)$ . The uniqueness built into this dilation theorem leads to a certain uniqueness with such a complementary pair, so that we may talk of “the” complementary channel  $\Phi^\sharp$  for a given channel  $\Phi$  [16, 17].

We have already discussed operator private subsystems—the essential difference being that instead of a single state on  $A$  as in Definition 2, it is demanded that Eq. (5) holds for all states on  $A$ . Similarly, an operator quantum error-correcting subsystem  $B$  for a channel  $\mathcal{E}$  [12, 13] requires the existence of a correction operation  $\mathcal{R}$  such that:  $\forall \sigma_A \forall \sigma_B, \exists \tau_A$  for which  $\mathcal{R} \circ \mathcal{E}(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B$ . The main result of [8] shows that  $B$  is private for  $\Phi$  if and only if it is error-correcting for  $\Phi^\sharp$ .

Does this result extend to the more general setting? The Kraus operators of the complementary map  $\Lambda^\sharp$  are four orthogonal rank-one projectors in two-qubit Hilbert space. No error-correcting subsystem can be extracted

in such a setting, moreover, when the input space is restricted to be that of our example, the complementary map is in fact private. Thus, not only does the complementarity result fail, it fails dramatically. We save calculations and further discussion on this point for [10].

Perhaps the notion of an error-correcting subsystem can be altered to mimic our new notion of a private quantum subsystem. A revised definition analogous to that of Def. 2 could be proposed:  $B$  is *correctable* for  $\mathcal{E}$  if there exists an operation  $\mathcal{R}$  such that for all  $\sigma_B$  and some *fixed* states  $\sigma_A, \tau_A$ , we have  $\mathcal{R} \circ \mathcal{E}(\sigma_A \otimes \sigma_B) = \tau_A \otimes \sigma_B$ . Yet, even under this new definition of error-correctable, the complementary channel  $\Lambda^\sharp$  cannot exhibit a correctable subsystem as the set of Kraus operators project onto a class of states that are insufficient to encode a qubit of information. As such, the only possibility for a complementarity result analogous to [8] is through a revised notion of the complementary channel.

### CONCLUSION AND OUTLOOK

We have studied a general notion of private quantum subsystems that encompasses all previously considered instances as special cases. Taking motivation from quantum error correction, specifically the Knill-Laflamme conditions, we presented algebraic conditions that characterize when a code is private for a given channel. We analyzed the development of private subsystems for the special case given by the composition of phase damping channels on many qubit Hilbert spaces. While each individual channel of this form is not private, the composition of such channels were shown to contain a private single qubit subsystem. Yet, for such classes of channels no private subspace or operator private subsystem exists. Moreover, we discussed how the channel fails to have the corresponding complementary error-correctable pair as in the case of operator subsystems. Nevertheless, this analysis naturally led us to define a potentially new form of quantum error-correcting code and warrants, along with the topic of channel complementarity, further investigation. And finally, preliminary discussions suggest there may be yet unexplored connections between the study of private quantum subsystems and notions of privacy in classical communication. In [10] and elsewhere we will continue and expand on the work initiated here.

### ACKNOWLEDGMENTS

We are grateful to Robert Spekkens for interesting conversations. T. J.-O. was supported by an NSERC Graduate Scholarship. D.W.K. was supported by NSERC and Ontario ERA. R.L. was supported by NSERC, QuantumWorks, and Industry Canada. S.P. was supported by an NSERC Graduate Scholarship.

- 
- [1] A. Ambainis, M. Mosca, A. Tapp, and R. de Wolf, IEEE Symposium on Foundations of Computer Science (FOCS), 547–553 (2000).
  - [2] P. O. Boykin and V. Roychowdhury, Phys. Rev. A **67**, 042317 (2003).
  - [3] S. D. Bartlett, T. Rudolph, R. W. Spekkens, Phys. Rev. A **70**, 032307 (2004).
  - [4] S. D. Bartlett, P. Hayden, and R. W. Spekkens, Phys. Rev. A **72**, 052329 (2005).
  - [5] R. Cleve, D. Gottesman, H.-K. Lo, Phys. Rev. Lett. **83**, 648 (1999).
  - [6] C. Crepeau, D. Gottesman, A. Smith, in Proc. 34<sup>th</sup> Annual Symposium on Theory of Computing, 643 (ACM, Montreal, 2002).
  - [7] J. Bouda and M. Ziman, J. Phys. A: Math. Theor. **40**, 5415 (2007).
  - [8] D. Kretschmann, D. W. Kribs, and R. W. Spekkens, Phys. Rev. A **78**, 032330 (2008).
  - [9] A. Church, D. W. Kribs, R. Pereira, and S. Plosker, Quantum Inf. Comput. **11**, 774–783 (2011).
  - [10] T. Jochym-O'Connor, D. W. Kribs, R. Laflamme, and S. Plosker, in preparation.
  - [11] E. Knill and R. Laflamme, Phys. Rev. A **55** 2, 900–911 (1997).
  - [12] D. Kribs, R. Laflamme, and D. Poulin, Phys. Rev. Lett. **94**, 180501 (2005).
  - [13] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky, Quantum Inf. Comput. **6**, 382 (2006).
  - [14] M. A. Nielsen, and D. Poulin, Phys. Rev. A **75**, 064304 (2007).
  - [15] W. F. Stinespring, Proc. Amer. Math. Soc. **6**, 211 (1955).
  - [16] A. S. Holevo, Probability Theory and Applications **51**, 133 (2006).
  - [17] C. King, K. Matsumoto, M. Nathanson, and M. B. Ruskai, Markov Processes and Related Fields **13**, 391–423 (2007).